Using a First Order Logic to Verify That Some Set of Reals Has No Lesbegue Measure

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Abstract. This paper presents a formal proof of Vitali's theorem that not all sets of real numbers can have a Lebesgue measure, where the notion of "measure" is given very general and reasonable constraints. A careful examination of Vitali's proof identifies a set of axioms that are sufficient to prove Vitali's theorem, including a first-order theory of the reals as a complete, ordered field, "enough" sets of reals, and the axiom of choice. The main contribution of this paper is a positive demonstration that the axioms and inference rules in ACL2(r), a variant of ACL2 with support for nonstandard analysis, are sufficient to carry out this proof.

1 Introduction

The notion of Lebesgue measure, reviewed in section 2, generalizes the concept of length for sets of real numbers. It is a surprising result in real analysis that not all sets of real numbers can be adequately assigned a measure, provided that the notion of "measure" conforms to some reasonable constraints, such as the measure defined by Lebesgue. This paper presents a formal proof of this result, which is due to Vitali.

The formal proof is carried out in ACL2(r), a variant of ACL2 that offers support for nonstandard analysis. Although ACL2(r) adds built-in support for key concepts from nonstandard analysis, such as "classical" and "standard part", it retains the strengths and limitations of ACL2. In particular, it is a strictly first-order theory, with only limited support for quantifiers. One important logical feature of ACL2 and ACL2(r) is the introduction of constrained functions, which allows these theorem provers to reason about classes of functions, e.g., all continuous functions, in a strictly first-order setting. This is similar to the mathematical practice of reasoning about a generic continuous function, in order to establish a theorem that applies to all continuous functions. Moreover, ACL2 and ACL2(r) support a definition principle that allows the introduction of "choice" functions via a Skolem axiom. This powerful definitional principle works as follows. Let ϕ be a formula whose only free variables are v, x_1, x_2, ..., x_n. The Skolem axiom introducing f from ϕ with respect to v is

$$\phi \Rightarrow \textbf{let } v = f(x_1, x_2, \ldots, x_n) \textbf{ in } \phi$$

What this axiom states is that the function f can "choose" an appropriate v for a given x_1, x_2, ..., x_n, provided such a choice is at all possible. This principle was recently "strengthened" in ACL2, and the strong version of this principle takes the place of the Axiom of Choice in the formal proof of Vitali's Theorem. We do not believe that this proof could have been carried out using the original version of this definitional principle. Perhaps surprisingly, these two definitional principles are conservative in the logic of ACL2. This logic is precisely described in [7, 3].

The paper is organized as follows. In section 2, we review the notion of Lebesgue measure, and we discuss the properties that a "reasonable" measure should have. This is followed by a review of Vitali's theorem in section 3. This is followed by a more introspective consideration of Vitali's proof. In section 4, we consider the key logical arguments that make up Vitali's proof and demonstrate how these have been formalized in ACL2(r). Note that this paper is self-contained. In particular, we do not assume that the reader is intimately familiar with Lebesgue measure, nonstandard analysis, or ACL2(r). Rather, we present the necessary background as it is needed.

2 Lebesgue Measure

The length of an interval of real numbers is the difference of the endpoints of the interval. Intervals include open, closed, and half-open intervals, with their usual meaning: (a, b), $[a, b]$, $(a, b]$, and $[a, b)$. Lebesgue measure extends the notion of length to more complicated sets than intervals.

One way to extend this notion is to introduce some "infinite" reals. The extended reals add the "numbers" $+\infty$ and $-\infty$ to the set of reals. These numbers are introduced so that they generalize the arithmetic and comparison operations in the obvious way. For example, $x < +\infty$ for any x other than $+\infty$. Similarly, $x + \infty = \infty$ for any x other than $-\infty$; the sum $(+\infty) + (-\infty)$ is not defined.

The other way in which the Lebesgue measure $m(S)$ extends the notion of length is to consider the measure of sets S that are not intervals. Lebesgue developed this notion by considering the sum of the length of sets of intervals that completely cover S. The details of that construction are not necessary for the remainder of this paper, but the interested reader is referred to [9].

2.1 Properties of an Ideal Measure

Ideally, the Lebesgue measure m should have the following properties:

1. $m(S)$ is a nonnegative extended real number, for each set of real numbers;
2. for an interval I, $m(I) = \text{length}(I)$;
3. m is countably additive: if $\langle S_n \rangle$ is a sequence of disjoint sets for which m is defined, then $m\left(\bigcup S_n\right) = \sum m(S_n)$;
4. m is translation invariant: if S is a set of reals for which m is defined and r is a real number, let $S + r$ be the set $\{s + r \mid s \in S\}$. Then $m(S + r) = m(S)$.

5. m is finitely additive: if S_1 and S_2 are disjoint sets, for which m is defined, then $m(S_1 \cup S_2) = m(S_1) + m(S_2)$.
6. m is monotonic: if $S_1 \subseteq S_2$ are sets for which m is defined, then $m(S_1) \leq m(S_2)$.

The last two properties can be derived from the previous ones.

Although these properties seem quite reasonable, they are contradictory. For instance, if properties (2)-(4) hold, then using the axiom of choice, a set (called V below) of real numbers can be constructed that cannot have Lesbegue measure, thus property (1) is violated. Such sets are called non-measurable sets.

3 Vitali's Theorem

Given a set S, a σ-algebra is a set of subsets of S that is closed under complements relative to S, finite unions, and countable unions. That is, if $\mathcal{A}$ is a σ-algebra of subsets of the set S, then

$\emptyset \in \mathcal{A}$
if $A \in \mathcal{A}$, then $S - A \in \mathcal{A}$,
if $A \in \mathcal{A}$ and $B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$, and
if $\langle A_i \rangle$ is a sequence of sets in $\mathcal{A}$, then $\bigcup_{i=0}^{\infty} A_i \in \mathcal{A}$.

By DeMorgan's Laws, a σ-algebra is closed under finite and countable intersections.

In the sequel, let $\mathbb{Q}$ be the set of all rationals and $\mathbb{R}$ be the set of all reals.

Definition 1 (Vitali's Set V). *Let E be the equivalence relation defined by*

$$xEy \Leftrightarrow x, y \in [0, 1) \wedge x - y \in \mathbb{Q}.$$

By the Axiom of Choice, there is a set V that contains exactly one element from each equivalence class.

This definition of the set V is essentially due to Vitali [11], who also showed that V is not Lebesgue measurable.

Theorem 1. *If m is a countably additive, translation invariant measure defined on a σ-algebra containing the intervals and the set V (defined above), then $m([0, 1))$ is either 0 or infinite.*

Proof. $V \subseteq [0, 1)$ has the property that for each $x \in [0, 1)$ there is a unique $y \in V$ and a unique $q \in \mathbb{Q}$ such that $x = y + q$.

Consider two cases.

- Case 1. $m(V) = 0$.
  Since $[0,1) \subseteq \bigcup\{V+q \mid q \in \mathbb{Q}\}$,

$$\begin{aligned} m([0,1)) &\leq m\left(\bigcup\{V+q \mid q \in \mathbb{Q}\}\right) \\ &= \sum_{q \in \mathbb{Q}} m(V+q) \\ &= \sum_{q \in \mathbb{Q}} m(V) \\ &= \sum_{q \in \mathbb{Q}} 0 \\ &= 0. \end{aligned}$$

- Case 2. $m(V) > 0$.
  Since

$$\begin{aligned} m\left(\bigcup\{V+q \mid 0 \leq q < 1 \wedge q \in \mathbb{Q}\}\right) &= \sum_{q \in [0,1) \cap \mathbb{Q}} m(V+q) \\ &= \sum_{q \in [0,1) \cap \mathbb{Q}} m(V) \\ &= +\infty \end{aligned}$$

  and $\bigcup\{V+q \mid 0 \leq q < 1 \wedge q \in \mathbb{Q}\} \subseteq [0,2)$,

$$\begin{aligned} +\infty &= m\left(\bigcup\{V+q \mid 0 \leq q < 1 \wedge q \in \mathbb{Q}\}\right) \\ &\leq m([0,2)). \end{aligned}$$

  Thus

$$\begin{aligned} +\infty &= m([0,2)) \\ &= m([0,1)) + m([1,2)) \\ &= m([0,1)) + m([0,1)+1) \\ &= m([0,1)) + m([0,1)) \\ &= 2 \cdot m([0,1)) \end{aligned}$$

  and so $m([0,1)) = +\infty$.

Thus the set V cannot be Lebesque measurable, for otherwise

$$m([0,1)) \neq 1 = \text{length}([0,1)).$$

□

We emphasize that we have not developed Lebesgue measure. Peter Loeb[5] found a way to use nonstandard analysis to develop Lebesgue measure on the set of standard real numbers.

4 What Is Needed for the Proof

In order to carry out Vitali's proof, we must have a significant logical machinery in place. First of all, we must have a theory of the real numbers, at least enough to formalize the reals as a complete, ordered field. Second, we must be able to reason about sets of reals. A complete set theory is not necessary, however. Only enough set theory to construct and manipulate V is required. Finally, the construction of V depends on the Axiom of Choice, so something similar to it must be available. In this section, we show how the logical machinery of ACL2(r) addresses these requirements.

4.1 First-Order Theory of the Reals

ACL2(r) introduces the real numbers using nonstandard analysis. A full treatment of nonstandard analysis can be found in [8], and the formalization of nonstandard analysis in ACL2(r) in [4]. In the following paragraphs, we present a brief description of nonstandard analysis in ACL2(r), for the benefit of readers who are unfamiliar with either.

In nonstandard analysis, the integers are classified as either *standard* or *nonstandard*. All of the familiar integers happen to be *standard*; however, there is at least one nonstandard integer N. Necessarily, $\pm N$, $\pm(N \pm 1)$, $\ldots \pm(N \pm k)$ are all nonstandard for any standard integer k. But notice that if k is nonstandard, $N - k$ may well be standard, e.g., when $k = N$. A number is called *i-large* if its magnitude is larger than any standard integer. The notion of *i-large* captures in a formal sense the intuitive notion of an "infinite" integer. An important fact is that all algebraic properties of the integers hold among both the standard and nonstandard integers.

These notions are easily extended to the reals. There are i-large reals, such as N, $\sqrt{N}$, e^N, etc. Consequently, there are also non-zero reals with magnitude smaller than any standard real, such as $1/N$. Such reals are called *i-small* and correspond with the intuitive notion of "infinitesimal." Note that the only standard number that is also i-small is zero. A number that is not i-large is called *i-limited*. All standard numbers are i-limited, as are all i-small numbers, as is the sum of any two i-limited numbers. Note that the nonstandard integers are precisely the i-large integers.

Two numbers are considered *i-close* if their difference is i-small. We write $x \approx y$ to mean that x is i-close to y. Every i-limited number x is i-close to a standard real, so it can be written as $x = \sigma(x) + \epsilon$, where $\sigma(x)$ is standard and ϵ is i-small. We call $\sigma(x)$ the *standard part* of x. Note that an i-limited number x can be i-close to only *one* standard number, so standard part is well defined. We note that the standard part function takes the place of least upper bounds and completeness of the reals in many arguments in analysis. Specifically, we use this notion for summing infinite series.

While there is much more to nonstandard analysis in ACL2(r), the notions above will be sufficient for the remainder of this paper. Before moving on to other aspects of the proof, however, it is beneficial to address alternative viewpoints of nonstandard analysis. One viewpoint holds that "standard" is a new property of numbers, one that cannot be captured in regular analysis. In this viewpoint, operations such as + and − are effectively unchanged, and they still operate over the same set of reals as before. An alternative viewpoint is that nonstandard analysis *extends* the real number line by introducing both infinitely small and infinitely large numbers, much in the same way that the reals can be constructed from the rationals.

These two viewpoints can be reconciled. For clarity, we refer to the traditional reals as "the reals" and the extended reals as "the hyperreals". In the first viewpoint, the set $\mathbb{R}$ corresponds to the hyperreals, and the reals are the elements of $\mathbb{R}$ that happen to be standard. In the second viewpoint, the set $\mathbb{R}$ denotes the reals (as usual), which are all standard, and the hyperreals are denoted by $\mathbb{R}^*$. ACL2(r) adopts the first viewpoint, so the predicate `(REALP X)` is true for any hyperreal `X`.

4.2 Sets of Reals

In this section we describe how we represent sets of reals in ACL2(r).

Some sets are represented by designated ACL2(r) λ-expressions. Each such λ-expression, Λ, represents the set of all ACL2(r) objects, a, for which the λ-application $(\Lambda\ a)$ evaluates to an ACL2(r) object other than `NIL`.

For example, the empty set $\emptyset$ is represented by the ACL2(r) λ-expression

```
(LAMBDA (X) (NOT (EQUAL X X))).
```

and the interval $[0, 1)$ is represented by the expression

```
(LAMBDA (X)
       ( (LAMBDA (A B X)
                 (AND (REALP X)
                      (AND (<= A X)
                           (<  X B))))
         '0
         '1
          X )).
```

Note that these are simply ACL2(r) literal constants—not functions. To treat them as functions, they are passed as arguments to an evaluator that mimics their execution. Evaluators can be defined in ACL2 and ACL2(r) for any finite set of previously defined functions. We defined an evaluator that knows about functions such as `and`, `+`, `realp`, `<`, and more specialized functions such as `Vp` and `Seq1` which are useful in the construction of V.

Once the evaluator is defined, it is almost mechanical to define functions to test for set membership, union, intersection, etc. The test for membership calls the evaluator, while the set operations manipulate the λ-expressions. Note that these functions operate over the hyperreals, since ACL2(r) interprets `REALP` as a recognizer for the hyperreals. However, parts of the argument are restricted to the reals, so we also defined membership predicates and set operations that interpret the expressions as ranging only over the (standard) reals[1].

We mention in passing that the definition of the set equality and subset predicates make use of the ACL2's limited support for quantification. This support is based on the idea of Skolem functions (which find either a witness or a counterexample to model $\exists$ and $\forall$). We will discuss these functions later, in the context of the Axiom of Choice.

We turn our attention now to the definition of countably infinite unions and intersections. Let Φ be a λ-expression, defining a unary function, whose domain includes the standard nonnegative integers, and whose range consists of λ-expressions, like those discussed above, that define sets. Then Φ enumerates a countable collection of sets. Let $\bigcup \Phi$ be the union, over all standard nonnegative integers, k, of the sets represented by the evaluation of the λ-applications, $(\Phi\ k)$.

We can represent $\bigcup \Phi$ in our set notation as follows. First, consider $\bigcup_n \Phi$, defined as the union of the sets $(\Phi\ k)$ for integer k up to n. Then $\bigcup_n \Phi$ can be represented as a λ-expression that encodes a disjunction of the $(\Phi\ k)$. Now suppose that n is an i-large integer. Then $\bigcup_n \Phi$ amounts to a λ-expression containing an "infinite" disjunction, and this is what we use to represent $\bigcup \Phi$. In particular, let $\mathbb{M}$ be the set of `standard` nonnegative integers. Then ACL2(r) can prove that for standard x, x is a member of $\bigcup \Phi$ if and only if $\exists k \in \mathbb{M}$ such that x is a member of the evaluation of the application $(\Phi\ k)$.

Notice that we don't quite have a σ-algebra of sets: We cannot form "infinite" unions of "infinite" unions. That is "infinite" unions are not allowed to be in the range of the λ-expression Φ. Nevertheless, enough sets exist to carry out the proof.

Since ACL2(r) is first-order, it is not possible in ACL2(r) to quantify over arbitrary σ-algebras and measures. However, we do have the ability to refer explicitly to many sets, including intervals and the set V, via their λ-expression definitions. Moreover, set operations including translations, unions, and even countable unions can all be done by manipulating such definitions.

Recall that ACL2(r) allows constrained functions. So a function m can be consistently constrained to satisfy versions of the required measure axioms that explicitly refer to λ definitions of sets.

For example, m can be constrained to satisfy axioms such as

- If S is the λ definition of a set of (standard) real numbers, then $m(S)$ is a nonnegative extended (standard) real.
- m is finitely and countably additive for definable sets of standard reals.
- m is translation invariant on definable sets of standard reals.

This means that the version of Theorem 1 that we actually prove can be formally stated as follows:

[1] Readers familiar with ACL2(r) may note that evaluators cannot be defined over the function `standardp`, due to limitations regarding recursion in the current version of ACL2(r). Defining different set operations so that `REALP` can be interpreted as a recognizer either for the reals or hyperreals solves this difficulty.

Theorem 2. *If m is a finitely and countably additive, translation invariant measure defined on a σ-algebra containing the λ-definable sets of standard reals, then $m([0, 1))$ is either 0 or infinite.*

Since m is constrained, we cannot explicitly state the theorem in ACL2(r), but we can, indeed, carry out the proof!

4.3 The Axiom of Choice

The Axiom of Choice postulates[6]: For every set S of nonempty sets, there is a function f such that for each set $s \in S$, $f(s) \in s$. Such a function f is called a choice function for S.

ACL2(r) implements first-order quantification by axiomatizing Skolem functions. That is, by suitably generalizing the following: $\exists y \varphi(x, y)$ is defined to be $\varphi(x, f(x))$, where x is the free variable in $\exists y \varphi(x, y)$ and the Skolem function f is a new function symbol satisfying the new Skolem axiom $\varphi(x, y) \rightarrow \varphi(x, f(x))$. The Skolem axiom means that the Skolem function can be viewed as a choice function: $f(x)$ chooses a value so that $\varphi(x, f(x)$ will be true, if such a value exists.

ACL2(r) explicitly implements Skolem functions as choice functions. One application of choice functions is the selection of a canonical element from each member of an equivalence class, as is done in Vitali's definition of the set V. However, before ACL2 version 3.1, this was not possible in ACL2.

Some of the discussions at recent ACL2 Workshops centered around this limitation of ACL2, and ACL2 was modified as a result of these discussions. To understand the precise limitation, consider an equivalence relation `E`. The following ACL2 event picks an equivalent `y` for each `x`:

```
(defchoose E-selector-weak (x) (y)
  (E x y))
```

So `(E-selector-weak x)` is always `E`-equivalent to `x`. However, suppose that `x1` is `E`-equivalent to `x2`. Then ACL2 does not guarantee any relationship between `(E-selector-weak x1)` and `(E-selector-weak x2)`. Hence `E-selector-weak` cannot be used to select a canonical member from each `E`-equivalence class.

The solution was to create a stronger `defchoose` function in ACL2, and this is done with the `:strengthen` keyword. In particular, the following `defchoose` *does* select a canonical element from each class:

```
(defchoose E-selector (x) (y)
  (E x y)
  :strengthen t)
```

Now, if `x1` is `E`-equivalent to `x2`, then `(E-selector x1)` is guaranteed to be equal to `(E-selector x2)`. So the range of `E-selector` is the set of canonical elements from the equivalence classes, as needed in the definition of Vitali's set V.

It would be easy to formalize the canonicalizing behavior of `E-selector` in a higher-order logic, but it requires a little care to do so in first-order logic. The `:strengthen` option formalizes this by adding one more constraint on the choice function `E-selector`. In particular, the `defchoose` function introduces the following constraining axioms:

- If there is any x such that $(E\, x\, y)$ is true, then $(E\, (\textit{E-selector}\, y)\, y)$ is also true.
- For all possible $y1$, at least one of the following must hold:
  - $(\textit{E-selector}\, y) = (\textit{E-selector}\, y1)$.
  - $(E\, (\textit{E-selector}\, y)\, y)$ is true, but $(E\, (\textit{E-selector}\, y)\, y1)$ is false.
  - $(E\, (\textit{E-selector}\, y1)\, y1)$ is true, but $(E\, (\textit{E-selector}\, y1)\, y)$ is false.

These axioms guarantee that `E-selector` chooses the same canonical value for each y any given equivalence class.

We do not believe that V could have been defined before the introduction of `:strengthen` into ACL2[2]. Specifically, we believe that without `:strengthen`, ACL2 does not have enough logical firepower to carry out many arguments that depend on the Axiom of Choice. However, even without `:strengthen`, ACL2 was able to prove some consequences of the Axiom of Choice that are strictly weaker than the Axiom. One of these is the Principle of Dependent Choices[6].

Definition 2 (Principle of Dependent Choices). *If ρ is a binary relation on a nonempty set S such that for every $x \in S$ there is a $y \in S$ with $x\rho y$, then there is a sequence $\langle x_n \rangle$ of elements from S such that $x_0 \rho x_1, x_1 \rho x_2, \ldots, x_n \rho x_{n+1}, \ldots$.*

We have proved a version of Dependent Choices in ACL2(r) just using the original Skolem axioms for choice functions, without the strengthening used to establish that choice functions can be made to select unique representatives from equivalence classes[1].

It is noteworthy, however, that Solovay[10] has shown that there is a model of set theory that satisfies the Principle of Dependent Choices, but in which every set of real numbers is Lebesgue measurable.

Moreover, the Principle of Dependent Choices is enough to make it possible to define a satisfactory Lebesgue measure[6]. Dependent Choices ensures, for example, that the set of all real numbers is not the countable union of countable sets and allows proofs of all the "positive" properties, desired by the analysts, of Lebesgue measure.

5 Conclusions

This paper described a formal proof of Vitali's Theorem. The proof depends on three pillars:

- A first-order theory of the reals, as provided by nonstandard analysis in ACL2(r).

[2] In fact, it was precisely this introduction that motivated our current work.

- A theory of sets sufficient to reason about countable unions of sets of reals. This theory could take many forms, but we chose a representation based on unary λ-expressions and an evaluator that interprets those expressions.
- The Axiom of Choice, which was simulated using Skolem choice functions in ACL2(r).

The third pillar is the most surprising, since it depends on the `:strengthen` feature of Skolem functions, which was only recently introduced into ACL2.

Figure 1 gives an idea of the effort to formalize Vitali's Theorem. The complete ACL2 proof scripts are available from the authors, and they will be added to the ACL2-Books Repository [2].

File	**Definitions**	**Theorems**	**Hints**
Analysis fundamentals	17	76	46
Extended reals	10	45	8
Enumeration of rationals	14	64	37
Set Support	50	146	70
Vitali's Construction	3	12	6
Vitali's Proof	0	81	57
Dependent choices	6	4	0

Fig. 1. Effort of work

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